Non-Hermitian d-dimensional Hamiltonians with position dependent mass and their η-pseudo-Hermiticity generators

Omar Mustafa[1] and S.Habib Mazharimousavi[2]
Department of Physics, Eastern Mediterranean University,
G Magusa, North Cyprus, Mersin 10,Turkey
[1]E-mail: omar.mustafa@emu.edu.tr
[2]E-mail: habib.mazhari@emu.edu.tr

February 1, 2008

Abstract

A class of non-Hermitian d-dimensional Hamiltonians with position dependent mass and their *η-pseudo-Hermiticity generators* is presented. Illustrative examples are given in 1D, 2D, and 3D for different position dependent mass settings.

PACS numbers: 03.65.Ge, 03.65.Fd,03.65.Ca

1 Introduction

The proposal of the non-Hermitian $\mathcal{PT}$ -symmetric Hamiltonians by Bender and Boettcher in [1] has relaxed the Hermiticity of a Hamiltonian as a necessary condition for the reality of the spectrum [1-7]. In the $\mathcal{PT}$ -symmetric setting, the hermiticity assumption $H = H^{\dagger}$ is replaced by the mere $\mathcal{PT}$ -symmetric one $\mathcal{PT}H\left(\mathcal{PT}\right)^{-1} = \mathcal{PT}H\mathcal{PT} = H$, where $\mathcal{P}$ denotes the parity $(\mathcal{P}x\mathcal{P} = -x)$ and the anti-linear operator $\mathcal{T}$ mimics the time reflection $(\mathcal{T}i\mathcal{T} = -i)$. Intensive attention was paid to the potentials $V(x)$ which are analytic on the full axis $x \in (-\infty, \infty)$ in one-dimension (1D).

However, in the transition to more dimensions with singularities manifested by, say, the central repulsive/attractive d-dimensional core, $\ell_d\left(\ell_d + 1\right)/r^2$ with $r \in (0, \infty)$, there still exist models unfortunate in methodical considerations. Nevertheless, a pioneering $\mathcal{PT}$ -symmetric model with physically acceptable impact has been the Buslaev and Grecchi [2] quartic anharmonic oscillator, where a simple constant downward shift of the radial coordinate (i.e., $r \rightarrow x - ic$; $x \in (-\infty, \infty)$ and $\operatorname{Im} r = -c < 0$) is employed (cf.,e.g., Znojil and Lévai [3] for illustrative examples).

In a broader class (where $\mathcal{PT}$ -symmetric Hamiltonians constitute a subclass among others) of non-Hermitian pseudo-Hermitian Hamiltonians [8-14] (a generalization of $\mathcal{PT}$ -symmetry, therefore), it is concreted that the eigenvalues of a pseudo-Hermitian Hamiltonian H are either real or come in complex-conjugate pairs. In this case, a Hamiltonian H is pseudo-Hermitian if it obeys the similarity transformation:

$$\eta\, H\, \eta^{-1} = H^{\dagger}$$

Where η is a Hermitian (and so is $\eta\, H$) invertible linear operator and $(^{\dagger})$ denotes the adjoint. However, the reality of the spectrum is secured if the pseudo-Hermitian Hamiltonian is an η-pseudo-Hermitian with respect to

$$\eta = O^{\dagger} O \tag{1}$$

for some linear invertible operator $O : \mathcal{H} \rightarrow \mathcal{H}$ (where $\mathcal{H}$ is the Hilbert space of the quantum system with a Hamiltonian H) and satisfies the intertwining relation (cf., e.g., [12,14] and references therein)

$$\eta\, H = H^{\dagger}\, \eta. \tag{2}$$

In a recent study [14], we have presented a class of spherically symmetric non-Hermitian Hamiltonians and their η-pseudo-Hermiticity generators. Therein, a generalization beyond the nodeless 1D states is proposed and illustrative examples are presented, including an exactly solvable non-Hermitian η-pseudo-Hermitian Morse model.

On the other hand, a position-dependent effective mass, $M\,(r) = m_{\circ} m\,(r)$, associated with a quantum mechanical particle constitutes a useful model for the study of many physical problems [15-28]. They are used, for example, in the energy density many-body problem [15], in the determination of the electronic properties of semiconductors [16] and quantum dots [17], in quantum liquids [18], in ^{3}He cluster [19], etc.

In this work, we present (in section 2) a d-dimensional recipe for *η-pseudo-Hermiticity generators* for a class of non-Hermitian Hamiltonians with position-dependent masses, $M\,(r) = m_{\circ} m\,(r)$. An immediate recovery of our generalized η-pseudo-Hermiticity generators for Hamiltonians with radial symmetry [14] is obvious through the substitution $m\,(r) = 1$. Our illustrative examples are given in section 3. Section 4 is devoted for our concluding remarks.

2 Non-Hermitian d-dimensional Hamiltonians with position dependent mass and their η-pseudo-Hermiticity generators

Following the symmetry ordering recipe of the momentum and position-dependent effective mass, $M(\vec{r}) = m_{\circ}\, m\,(\vec{r})$, a Schrödinger Hamiltonian with a complex potential field $V\,(\vec{r}) + iW\,(\vec{r})$ would read

$$H = \frac{1}{2}\left(\vec{p}\frac{1}{M\,(\vec{r})}\right)\cdot\vec{p} + V\,(\vec{r}) = -\frac{\hbar}{2m_{\circ}}\left(\vec{\nabla}\frac{1}{m\,(\vec{r})}\right)\cdot\vec{\nabla} + V\,(\vec{r}) + iW\,(\vec{r})\,. \tag{3}$$

Using the atomic units ($\hbar = m_\circ = 1$), and assuming the d-dimensional spherical symmetric recipe (cf, e.g., Mustafa and Znojil in [6]), with

$$\Psi\left(\vec{r}\right) = r^{-(d-1)/2} R_{n_r,\ell_d}\left(r\right) Y_{\ell_d,m_d}\left(\theta,\varphi\right), \tag{4}$$

Hamiltonian (3) would result in the following d-dimensional non-Hermitian Hamiltonian with a position-dependent mass $M\left(r\right) = m_\circ m\left(r\right)$ (in $\hbar = m_\circ = 1$ units)

$$H = -\frac{1}{2m\left(r\right)}\partial_r^2 + \frac{\ell_d\left(\ell_d+1\right)}{2m\left(r\right)r^2} - \frac{m'\left(r\right)}{2m\left(r\right)^2}\left(\frac{d-1}{2r} - \partial_r\right) + V\left(r\right) + iW\left(r\right). \tag{5}$$

Where $\ell_d = \ell + \left(d-3\right)/2$ for $d \geq 2$, ℓ is the regular angular momentum quantum number, $n_r = 0, 1, 2, \cdots$ is the radial quantum number, and $m'\left(r\right) = dm\left(r\right)/dr$. Moreover, the $d = 1$ can be obtained through $\ell_d = -1$ and $\ell_d = 0$ for even and odd parity, $\mathcal{P} = \left(-1\right)^{\ell_d+1}$, respectively.

Then H has a real spectrum if and only if there is an invertible linear operator $O : \mathcal{H} \longrightarrow \mathcal{H}$ such that H is η-pseudo-Hermitian with the linear invertible operator

$$O = \mu\left(r\right)\partial_r + Z(r) \implies O^\dagger = -\mu\left(r\right)\partial_r - \mu'\left(r\right) + Z^*(r) \tag{6}$$

where

$$\begin{gathered} Z(r) = F(r) + iG(r); \\ F(r) = \left[-\frac{\left(\ell_d+1\right)}{r} + f\left(r\right)\right]\mu\left(r\right), \quad G(r) = g\left(r\right)\mu\left(r\right) \end{gathered} \tag{7}$$

and $\mu\left(r\right) = \sqrt{1/2m\left(r\right)}$, $f(r)$, $g(r)$ are real-valued functions and $\mathbb{R} \ni r \in (0, \infty)$. Equation (1), in turn, implies

$$\begin{aligned} \eta &= -\mu\left(r\right)^2\partial_r^2 - \left[2\mu\left(r\right)\mu'\left(r\right) + 2i\mu\left(r\right)G\left(r\right)\right]\partial_r \\ &\quad - \left[\mu\left(r\right)Z'(r) + \mu'\left(r\right)Z(r) - F(r)^2 - G\left(r\right)^2\right], \end{aligned} \tag{8}$$

where primes denote derivatives with respect to r. Herein, it should be noted that the operators O and $O^\dagger$ are two intertwining operators and the Hermitian operator η only plays the role of a certain auxiliary transformation of the dual Hilbert space and leads to the intertwining relation (2) (cf, e.g.,[12]). Hence, considering relation (2) along with the eigenvalue equation for the Hamiltonian, $H/E_i\rangle = E_i/E_i\rangle$, and its adjoint, $H^\dagger/E_i\rangle = E_i^*/E_i\rangle$, one can show that any two eigenvectors of H satisfy

$$\langle E_i/H^\dagger\eta - \eta H/E_j\rangle = 0 \implies \left(E_i^* - E_j\right)\langle\langle E_i/E_j\rangle\rangle_\eta = 0. \tag{9}$$

Which implies that if $E_i^* \neq E_j$ then $\langle\langle E_i/E_j\rangle\rangle_\eta = 0$. Therefore, the η- orthogonality of the eigenvectors suggests that if ψ is an eigenvector (of eigenvalue $E = E_1 + iE_2$, $\forall E_1, E_2 \in \mathbb{R}$) related to H then

$$\eta\psi = 0 \implies O^\dagger O\psi = 0 \implies O\psi = 0, \tag{10}$$

and

$$\frac{Z(r)}{\mu(r)}=-\frac{\psi'(r)}{\psi(r)}=-\partial_r\ln\psi(r)\implies\psi(r)=\exp\left(-\int^r\frac{Z(z)}{\mu(z)}dz\right).\tag{11}$$

The intertwining relation (2) would lead to

$$W(r)=-2\mu(r)G'(r)\tag{12}$$

$$V(r)=F(r)^2-G(r)^2-\mu'(r)F(r)-\mu(r)F'(r)+\beta\tag{13}$$

$$\begin{aligned}F(r)^2-\mu'(r)F(r)-\mu(r)F'(r)=&\frac{1}{2}\left[\frac{\mu(r)^2G''(r)}{G(r)}-\mu(r)\mu''(r)\right]\\&-\frac{\mu(r)^2}{4G(r)^2}\left[\left(\frac{G(r)}{\mu(r)}\right)'\right]^2+\frac{\alpha}{4G(r)^2}\end{aligned}\tag{14}$$

where $\alpha,\beta\in\mathbb{R}$ are integration constant.

On the other hand, with $E=E_1+iE_2$, H in (5), and $\psi(r)$ in (11) the eigenvalue problem $H\psi(r)=E\psi(r)$ implies

$$\beta=E_1\tag{15}$$

$$F(r)=\frac{\mu'(r)G(r)-\mu(r)G'(r)-E_2}{2G(r)}\tag{16}$$

and

$$\alpha=E_2^2\tag{17}$$

Obviously, one would accept $\mathbb{R}\ni\alpha\geq0\implies\mathbb{R}\ni E_2=\pm\sqrt{\alpha}$, and negate $\alpha<0\implies E_2\in\mathbb{C}$ since $\mathbb{R}\ni E_2\notin\mathbb{C}$, a requirement of pseudo-Hermiticity mentioned early on. Yet $E_2\in\mathbb{C}$ contradicts with the real/imaginary descendants, (12) to (17). However, the reality of the spectrum is secured by our η-pseudo-Hermitcity generators. This in turn acquires $\alpha=0$ in the forthcoming developments.

The Hamiltonian in (5) may now be recast as

$$H=-\frac{1}{2m(r)}\partial_r^2+\frac{m'(r)}{2m(r)^2}\partial_r+\tilde{V}(r)+iW(r)\tag{18}$$

where

$$\tilde{V}(r)=\frac{\ell_d(\ell_d+1)}{2m(r)r^2}-\frac{m'(r)}{2m(r)^2}\left(\frac{d-1}{2r}\right)+V(r).\tag{19}$$

We may now summarize our results in terms of our *η-pseudo-Hermiticity* generators $g(r)$ and $f(r)$ as

$$W(r)=-2\mu(r)\left[g(r)\mu(r)\right]'\tag{20}$$

$$\tilde{V}(r) = \frac{\ell_d(\ell_d+1)}{2m(r)r^2} + \frac{2\mu(r)\mu'(r)[\ell_d+1]}{r} + \mu(r)^2\left[f(r)^2 - g(r)^2\right]$$
$$-\frac{2(\ell_d+1)}{r}f(r)\mu(r)^2 - 2\mu'(r)\mu(r)f(r) - \mu(r)^2 f'(r) + \beta \qquad (21)$$

$$g(r) = r^{2(\ell_d+1)}\exp\left(-2\int^r f(z)dz\right) \qquad (22)$$

and

$$\psi(r) = r^{\ell_d+1}\exp\left(-\int^r [f(z)+ig(z)]\,dz\right) \qquad (23)$$

Hence, $f(r)$ and/or $g(r)$ are our generating function(s) for the *η-pseudo-Hermiticity* of the class of non-Hermitian Hamiltonians in (5) with real spectra and $\psi(r)$ in (25) as an eigenfunction (not necessarily normalizable). Nevertheless, it should be reported here that our results cover the 1D Fityo's ones [13] through the substitutions $\ell_d = -1$, $m(r) = 1/2$, and $r \in (0,\infty) \to x \in (-\infty,\infty)$. Moreover, they also collapse into our recent results on *η-pseudo-Hermiticity* generators in [14] by the substitutions $\ell_d = \ell$, and $m(r) = 1/2$ (where we considered constant mass settings).

3 Illustrative examples

In this section, we construct *η-pseudo-Herrmiticity* of some non-Hermitian Hamiltonians with position-dependent mass using the above mentioned procedure through the following illustrative examples:

Example 1: For a quantum particle endowed with a position-dependent mass $m(r) = r^2/2$ and with the generating function $f(r) = r$ we consider the following cases:

A) For $d = 3, \ell = 0$, and $\mathbb{R} \ni r \in (0,\infty)$, one finds

$$g(r) = r^2 e^{-r^2} \qquad (24)$$

$$\tilde{V}(r) = -\frac{1}{r^2} - \frac{2}{r^4} - r^2 e^{-2r^2} + 1 + \beta \qquad (25)$$

$$W(r) = -\frac{2}{r}e^{-r^2} + 4re^{-r^2} \qquad (26)$$

$$\psi(r) = \left(\frac{4}{\sqrt{\pi}}\right)^{1/2} r\exp\left(-\frac{r^2}{2} - i\left(\frac{-r}{2}e^{-r^2} + \frac{\sqrt{\pi}}{4}\operatorname{erf}(r)\right)\right) \qquad (27)$$

B) For $d = 3, \ell = 1$, and $\mathbb{R} \ni r \in (0, \infty)$:

$$g(r) = r^4 e^{-r^2} \tag{28}$$

$$\tilde{V}(r) = -\frac{3}{r^2} - \frac{2}{r^4} - r^6 e^{-2r^2} + 1 + \beta \tag{29}$$

$$W(r) = 2r\left(-3 + 2r^2\right) e^{-r^2} \tag{30}$$

$$\psi(r) = \left(\frac{8}{3\sqrt{\pi}}\right)^{1/2} r^2\, e^{-r^2/2} \exp\left(-i\left(-\frac{r^3}{2}e^{-r^2} - \frac{3r}{4}e^{-r^2} + \frac{3\sqrt{\pi}}{8}\operatorname{erf}(r)\right)\right) \tag{31}$$

C) For $d = 2, \ell = 0$ and $\mathbb{R} \ni r \in (0, \infty) \to \mathbb{R} \ni \rho \in (0, \infty)$:

$$g(\rho) = \rho e^{(-\rho^2)} \tag{32}$$

$$\tilde{V}(\rho) = -\frac{5}{4}\frac{1}{\rho^4} - e^{-2\rho^2} + 1 + \beta \tag{33}$$

$$W(\rho) = 4e^{-\rho^2} \tag{34}$$

$$\psi(r) = \sqrt{2}\sqrt{\rho}\exp\left(-\frac{\rho^2}{2} - i\left(\frac{\rho}{2}e^{-\rho^2}\right)\right) \tag{35}$$

D) For $d = 1, \ell = 0$ and $\mathbb{R} \ni r \in (0, \infty) \to \mathbb{R} \ni x \in (-\infty, \infty)$;

$$g(x) = e^{(-x^2)} \tag{36}$$

$$\tilde{V}(x) = \frac{1}{x^2} - \frac{e^{-2x^2}}{x^2} + 1 + \beta \tag{37}$$

$$W(x) = 4\frac{e^{-x^2}}{x} + \frac{2e^{-x^2}}{x^3} \tag{38}$$

$$\psi(x) = \left(\frac{1}{\sqrt{\pi}}\right)^{1/2} \exp\left(-\frac{x^2}{2} - i\left(\frac{x}{2}\sqrt{\pi}\operatorname{erf}(x)\right)\right) \tag{39}$$

Example 2: For a quantum particle endowed with a position-dependent mass $m(r) = 1/\left[2\cosh^2(r)\right]$ and with the generating function $f(r) = \tanh(r)/2$, we consider the following cases:

i) For $d=1, \ell=0$ and $\mathbb{R} \ni r \in (0,\infty) \to \mathbb{R} \ni x \in (-\infty,\infty)$ one gets

$$g(x) = \frac{1}{\cosh(x)} \tag{40}$$

$$\tilde{V}(x) = -\frac{3\cosh^2(x)}{4} - \frac{3}{4} + \beta \tag{41}$$

$$W(x) = 0 \tag{42}$$

$$\psi(r) = \sqrt{\frac{1}{\sqrt{\pi}}} \frac{1}{\sqrt{\cosh(x)}} \exp\left(-2i \tanh^{-1}(e^x)\right) \tag{43}$$

ii) For $d=2, \ell=0$ and $\mathbb{R} \ni r \in (0,\infty) \to \mathbb{R} \ni \rho \in (0,\infty)$:

$$g(\rho) = \frac{\rho}{\cosh(\rho)} \tag{44}$$

$$\tilde{V}(\rho) = -\rho^2 - \frac{\cosh^2(\rho)}{4\rho^2} - \frac{3}{4}\cosh^2(\rho) + \frac{\cosh(\rho)\sinh(\rho)}{2\rho} + \frac{1}{4} + \beta \tag{45}$$

$$W(\rho) = -2\cosh(\rho) \tag{46}$$

$$\psi(\rho) = \left(2/\sqrt{\pi}\right)^{1/2} \sqrt{\frac{\rho}{\cosh(\rho)}} \exp\left(-i\int^{\rho} \frac{z}{\cosh(z)} dz\right) \tag{47}$$

iii) For $d=3, \ell=0$ and $\mathbb{R} \ni r \in (0,\infty)$:

$$g(r) = \frac{r^2}{\cosh(r)} \tag{48}$$

$$\tilde{V}(r) = -r^4 - \frac{3}{4}\cosh^2(r) + \frac{\cosh(r)\sinh(r)}{r} + \frac{1}{4} + \beta \tag{49}$$

$$W(r) = -4\cosh(r) \tag{50}$$

$$\psi(r) = (0.5079)\sqrt{\frac{r^2}{\cosh(r)}} \exp\left(-i\int^{r} \frac{z^2}{\cosh(z)} dz\right) \tag{51}$$

iv) For $d=3, \ell=2$ and $\mathbb{R} \ni r \in (0,\infty)$:

$$g(r) = \frac{r^6}{\cosh(r)} \tag{52}$$

$$\tilde{V}(r) = -r^{12} - \frac{3}{4}\cosh^2(r) + \frac{3\sinh(2r)}{2r} + \frac{6\cosh^2(r)}{r^2} + \frac{1}{4} + \beta \tag{53}$$

$$W(r) = -12r^5\cosh(r) \tag{54}$$

$$\psi(r) = (0.02636)\; r^3\sqrt{\frac{1}{\cosh(r)}}\exp\left(-i\int^r \frac{z^6}{\cosh(z)}dz\right) \tag{55}$$

4 Concluding Remarks

In this paper a class of non-Hermitian d-dimensional Hamiltonians for particles endowed with position-dependent mass and their η-*pseudo-Hermiticity generators* is introduced. Our illustrative examples include the 1D, 2D, and 3D Hamiltonians at different position-dependent mass settings. The results are presented in such a way that they reproduce our generalized η-*pseudo-Hermiticity generators* reported in [14] for node-less states with $m(r) = 1$ and $\ell_d = \ell$. The 1D Fityo's [13] results are also reproducible through the substitutions $\ell_d = -1$, $m(r) = 1$, and $r \in (0,\infty) \to x \in (-\infty,\infty)$. It should be noted, moreover, that example 2-i, with the imaginary part of the effective potential $W(x) = 0$, documents that Hermiticity is a possible by-product of η-*pseudo-Hermiticity* .

It is anticipated that with some luck one would be able to obtain some position-dependent mass non-Hermitian η-*pseudo-Hermitian Hamiltonians* that are exactly solvable. However, this issue already lies beyond our current methodical proposal. Moreover, a point canonical tansformation could be invested in the process to serve for exact solvability, an issue we shall deal with in the near future.

References

[1] C. M. Bender and S. Boettcher: Phys. Rev. Lett. **24** (1998) 5243

C. M. Bender, S. Boettcher and P.N. Meisinger: J. Math. Phys. **40** (1999) 2201

B. Bagchi, F. Cannata and C. Quesne: Phys. Lett. **A 269** (2000) 79

[2] V. Buslaev and V. Grecchi: J. Phys. A: Math. Gen. **26** (1993) 5541

[3] M. Znojil and G. Lévai: Phys. Lett. A **271** (2000) 327

[4] Z. Ahmed: Phys. Lett. **A 286** (2001) 231

B. Bagchi, S. Mallik, C. Quesne and R. Roychoudhury: Phys. Lett. **A 289** (2001) 34

[5] P. Dorey, C. Dunning and R. Tateo: J. Phys. A: Math. Gen. **4** (2001) 5679

R. Kretschmer and L. Szymanowski: Czech. J.Phys **54** (2004) 71

[6] M. Znojil, F. Gemperle and O. Mustafa: J. Phys. A: Math. Gen. **35** (2002) 5781

O. Mustafa and M. Znojil: J. Phys. A: Math. Gen. **35** (2002) 8929

[7] O. Mustafa: J. Phys. A: Math. Gen. **36** (2003) 5067

F. Fernandez, R. Guardiola, J. Ros and M. Znojil: J. Phys. A: Math. Gen. **31** (1998) 10105

[8] A. Mostafazadeh: J. Math. Phys. **43** (2002) 205

A. Sinha and P. Roy: Czech. J. Phys. **54** (2004) 129

[9] A. Mostafazadeh: J. Math. Phys. **43** (2002) 3944

A. Mostafazadeh: J. Math. Phys. **44** (2003) 974

[10] L. Jiang, L. Z. Yi and C.S. Jia: Phys Lett **A 345** (2005) 279

B.P. Mandal: Mod. Phys. Lett. **A 20** (2005) 655

[11] M. Znojil, H. Bíla and V. Jakubsky: Czech. J. Phys. **54** (2004) 1143

A. Mostafazadeh and A. Batal: J. Phys.A: Math. Gen. **37** (2004) 11645

[12] A. Mostafazadeh: J. Math. Phys. **43** (2002) 2814

A. Mostafazadeh: Nucl.Phys. **B 640** (2002) 419

A. Mostafazadeh: J. Phys. A: Math. Gen. **38** (2005) 3213

[13] T. V. Fityo: J. Phys. A: Math. Gen. **35** (2002) 5893

[14] O. Mustafa and S.H. Mazharimousavi: "Generalized η-pseudo-Hermiticity generators; radially symmetric Hamiltonians" (2006) (arXiv: hep-th/0601017)

[15] A. Puente and M. Casas: Comput. Mater Sci. **2** (1994) 441

[16] G. Bastard: *"Wave Mechanics Applied to Semiconductor Heterostructures"* , (1988) Les Editions de Physique, Les Ulis

[17] L.I. Serra and E. Lipparini: Europhys. Lett. **40** (1997) 667

[18] F. Arias de Saaverda, J. Boronat, A. Polls and A. Fabrocini: Phys. Rev. **B** **50** (1994) 4248

[19] M. Barranco et. al: Phys. Rev. **B** **56** (1997) 8997

[20] A. Puente, L.I. Serra and M. Casas: Z. Phys. **D** **31** (1994) 283

[21] A. R. Plastino, M. Casas and A. Plastino: Phys. Lett. **A** **281** (2001) 297

[22] O. Mustafa and S.H. Mazharimousavi: "Point canonical transformation d-dimensional regularization" (2006) (arXiv: math-ph/0602044)

O. Mustafa and S.H. Mazharimousavi: "Quantum particles trapped in a position-dependent mass barrier; a d-dimensional recipe " (2006) (arXiv: quant-ph/0603134)

[23] S. H. Dong and M. C. Lozada: Phys. Lett. **A** **337** (2005) 313

I.O. Vakarchuk: J. Phys. **A**; Math and Gen **38** (2005) 4727

C.Y. Cai, Z.Z. Ren and G.X. Ju: Commun. Theor. Phys. **43** (2005) 1019

[24] B. Bagchi, A. Banerjee, C. Quesne and V.M. Tkachuk: J. Phys. **A**; Math and Gen **38** (2005) 2929

J. Yu and S.H. Dong: Phys. Lett. **A** **325** (2004) 194

L. Dekar, L. Chetouani and T.F. Hammann: J. Math. Phys. **39** (1998) 2551

[25] C. Quesne: Ann. Phys. **321** (2006) 1221

C. Quesne and V.M. Tkachuk: J. Phys. **A**; Math and Gen **37** (2004) 4267

T. Tanaka: J. Phys. **A**; Math and Gen **39** (2006) 219

L. Jiang, L.Z. Yi, and C.S. Jia: Phys. Lett. **A** **345** (2005) 279

A.D. Alhaidari: Int. J. Theor. Phys. **42** (2003) 2999

[26] A.D. Alhaidari: Phys. Rev. **A** **66** (2002) 042116

[27] R. De, R. Dutt and U. Sukhatme: J. Phys. **A**; Math and Gen **25** (1992) L843

[28] G. Junker: J. Phys. **A**; Math and Gen **23** (1990) L881